A FIXED POINT APPROACH TO STABILITY OF A QUADRATIC EQUATION

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Abstract. Using the fixed point alternative theorem we establish the orthogonal stability of quadratic functional equation of Pexider type $f(x+y)+g(x-y)=h(x)+k(y)$, where f,g,h,k are mappings from a symmetric orthogonality space to a Banach space, by orthogonal additive mappings under a necessary and sufficient condition on f.

1. Introduction.

Suppose that $\mathcal{X}$ is a real vector space with $\dim \mathcal{X} \geq 2$ and $\perp$ is a binary relation on $\mathcal{X}$ with the following properties:

(O1) *totality of* $\perp$ *for zero*: $x \perp 0, 0 \perp x$ for all $x \in \mathcal{X}$;

(O2) *independence*: if $x, y \in \mathcal{X} - \{0\}, x \perp y$, then x, y are linearly independent;

(O3) *homogeneity*: if $x, y \in \mathcal{X}, x \perp y$, then $\alpha x \perp \beta y$ for all α, β in the real line $\mathbb{R}$;

(O4) *the Thalesian property*: Let P be a 2-dimensional subspace of $\mathcal{X}$. If $x \in P$ and λ in the nonnegative real numbers $\mathbb{R}_+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x+y_0 \perp \lambda x - y_0$.

Then the pair $(\mathcal{X}, \perp)$ is called an *orthogonality space*; cf. [25]. By an *orthogonality normed space* we mean an orthogonality space equipped with a norm. Some examples of special interest are

(i) The trivial orthogonality on a vector space $\mathcal{X}$ defined by (O1), and for non-zero elements $x, y \in \mathcal{X}$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(\mathcal{X}, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

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(iii) The Birkhoff-James orthogonality on a normed space $(\mathcal{X}, \|.\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$; cf. [15].

The relation $\perp$ is called *symmetric* if $x \perp y$ implies that $y \perp x$ for all $x, y \in \mathcal{X}$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than or equal to 3 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric; see [2].

Let $\mathcal{X}$ be a vector space (an orthogonality space) and $(\mathcal{G}, +)$ be an abelian group. A mapping $f : \mathcal{X} \to \mathcal{G}$ is called *(orthogonally) additive* if it satisfies the so-called *(orthogonal) additive functional equation* $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathcal{X}$ (with $x \perp y$). A mapping $f : \mathcal{X} \to \mathcal{G}$ is said to be *(orthogonally) quadratic* if it satisfies the so-called *(orthogonally) Jordan-von Neumann quadratic functional equation* $f(x+y)+f(x-y) = 2f(x)+2f(y)$ for all $x, y \in \mathcal{X}$ (with $x \perp y$).

The problem of "stability of functional equations" is that "when the solutions of an equation differing slightly from a given one must be close to an exact solution of the given equation?". In 1941, S. M. Ulam [28] posed the first question on the subject concerning the stability of group homomorphisms. In 1941, D. H. Hyers [12] gave a partial solution of Ulam's problem in the context of Banach spaces. In 1978, Th. M. Rassias [23] generalized the theorem of Hyers to an unbounded situation. The result of Rassias has provided a lot of influence in the development of what we now call *Hyers–Ulam–Rassias stability* of functional equations. Following Hyers and Rassias approaches, during the last decades, the stability problem for several functional equations have been extensively investigated by many mathematicians; cf. [13]. Nowadays, there may be found several applications in actuarial and financial mathematics, sociology, psychology, and pure mathematics [1].

The first author who treated the stability of the quadratic equation was F. Skof [26]. P. W. Cholewa [3] extended Skofs theorem to abelian groups. Skof's result was also generalized by S. Czerwik [5] in the spirit of Hyers–Ulam–Rassias. S. M. Jung [17, 18] investigated the stability of the quadratic equation. K.W. Jun and Y. H. Lee [16] proved the stability of quadratic equation of Pexider type. The stability problem of the quadratic equation has been extensively investigated by some mathematicians; cf. [6, 7, 24].

The orthogonal quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y),\ x\perp y$$

was first investigated by F. Vajzović [29] when $\mathcal{X}$ is a Hilbert space, $\mathcal{G}$ is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. H. Drljević [9] proved the following stability result:

Let $\mathcal{H}$ be a complex Hilbert space of dimension ≥ 3, and $A:\mathcal{H}\to\mathcal{H}$ a bounded self-adjoint linear operator with $\dim A(\mathcal{H})\geq 2$, and let the real numbers $\theta\geq 0$ and $p\in[0,2)$ be given. Suppose that $f:\mathcal{H}\to\mathbb{C}$ is continuous and satisfies the inequality

$$|f(x+y)+f(x-y)-2f(x)-2f(y)|\leq\theta[|\langle x,x\rangle|^{p/2}+|\langle y,y\rangle|^{p/2}],$$

whenever $\langle Ax,y\rangle=0$. Then the limit $T(x)=\lim_{n\to\infty}\frac{f(2^nx)}{4^n}$ exists for each $x\in\mathcal{H}$ and the functional T is continuous and satisfies $T(x+y)+T(x-y)=2T(x)+2T(y)$ whenever $\langle Ax,y\rangle=0$. Moreover, there exists a real number $\varepsilon>0$ such that

$$|f(x)-T(x)|\leq\varepsilon|\langle Ax,x\rangle|^{p/2},$$

for all $x\in\mathcal{H}$.

Later H. Drljević [8], M. Fochi [10] and G. Szabó [27] obtained more results on the subject.

One of the significant conditional equations is the so-called *orthogonally quadratic functional equation of Pexider type* $f(x+y)+g(x-y)=h(x)+k(y),\ x\perp y$. Recently, the second author investigated this equation with "$g=f$". Using some ideas from [11, 19, 21, 22, 20, 4], we aim to use the alternative of fixed point theorem to establish the stability of this equation in the spirit of Hyers–Ulam under certain conditions. The first systematic study of fixed point theorems in nonlinear analysis is due to G. Isac and Th. M. Rassias; cf. [14].

2. Main Results.

We start our work with a known fixed point theorem which will be needed later:

Theorem 2.1. *(The alternative of fixed point) Suppose $(\mathcal{E},d)$ be a complete generalized metric space and $J:\mathcal{E}\to\mathcal{E}$ be a strictly contractive mapping with the Lipschitz constant L. Then, for each given element $x\in\mathcal{E}$, either*

(A1) $d(J^nx,J^{n+1}x)=\infty$

for all $n \geq 0$, *or*

(A2) There exists a natural number n_0 *such that:*

(A20) $d(J^n x, J^{n+1}x) < \infty$, *for all* $n \geq n_0$*;*

(A21) The sequence $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J*;*

(A22) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in \mathcal{E} : d(J^{n_0}x, y) < \infty\}$*;*

(A23) $d(y, y^*) < \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

Suppose that $\mathcal{X}$ denotes an orthogonality real space and $\mathcal{Y}$ denotes a Banach space. Consider the set $\mathcal{E} := \{\varphi : \mathcal{X} \to \mathcal{Y} : \varphi(0) = 0\}$ and introduce a generalized metric on $\mathcal{E}$ by

$$d(\varphi, \psi) = \inf\{c \in (0, \infty) : \|\varphi(x) - \psi(x)\| \leq c, \forall x \in \mathcal{X}\}.$$

It is easy to see that $(\mathcal{E}, d)$ is complete. Given a number $0 \leq \lambda < 1$, define the following mapping $J_\lambda : \mathcal{E} \to \mathcal{E}$ by $(J_\lambda \varphi)(x) := \lambda \varphi(2x)$. For arbitrary elements $\varphi, \psi \in \mathcal{E}$ we have

$$\begin{aligned} d(\varphi, \psi) < c \;&\Rightarrow\; \|\varphi(x) - \psi(x)\| \leq c,\ x \in \mathcal{X} \\ &\Rightarrow\; \|\lambda\varphi(2x) - \lambda\psi(2x)\| \leq \lambda c,\ x \in \mathcal{X} \\ &\Rightarrow\; d(J_\lambda\varphi, J_\lambda\psi) \leq \lambda c. \end{aligned}$$

Therefore

$$d(J_\lambda\varphi, J_\lambda\psi) \leq \lambda d(\varphi, \psi),\ \varphi, \psi \in \mathcal{E}.$$

Hence J_λ is a strictly contractive mapping on $\mathcal{E}$ with the Lipschitz constant λ and we can use the fixed point alternative theorem.

We are just ready to prove the orthogonal stability of the Pexiderized equation $f(x+y) + g(x+y) = h(x) + k(y)$ where f, g, h, k are mappings from $\mathcal{X}$ to $\mathcal{Y}$ under certain condition.

We use the notation $\varphi(x) \trianglelefteq \varepsilon$ in the sense that there exists a number a such that $\varphi(x) \leq a\varepsilon$ for all x in the domain of φ.

Theorem 2.2. *Suppose that* $\mathcal{X}$ *is a real orthogonality space with a symmetric orthogonal relation* $\perp$ *and* $\mathcal{Y}$ *is a Banach space. Let the mappings* $f, g, h, k : \mathcal{X} \to \mathcal{Y}$ *satisfy the following inequalities*

$$\|f(x+y) + g(x-y) - h(x) - k(y)\| \leq \varepsilon, \tag{2.1}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. Then there exists an orthogonally additive mapping T such that

$$\|f(x) - T(x)\| \trianglelefteq \varepsilon$$

if and only if

$$\|f(2x) - f(-2x) - 4f(x) - 4f(-x)\| \trianglelefteq \varepsilon.$$

Indeed, if

$$\|f(2x) - f(-2x) - 4f(x) - 4f(-x)\| \leq \varepsilon. \tag{2.2}$$

holds for all $x \in \mathcal{X}$, then there exist orthogonally additive mappings $T, T', T'' : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{140}{3}\varepsilon,$$

$$\|g(x) - g(0) - T'(x)\| \leq \frac{98}{3}\varepsilon,$$

$$\|h(x) + k(x) - h(0) - k(0) - T''(x)\| \leq \frac{256}{3}\varepsilon,$$

for all $x \in \mathcal{X}$.

Proof. Suppose that (2.2) holds. Define $F(x) = f(x) - f(0), G(x) = g(x) - g(0), H(x) = h(x) - h(0), K(x) = k(x) - k(0)$. Then $F(0) = G(0) = H(0) = K(0) = 0$. Set $L(x) = \frac{H(x)+K(x)}{2}$.

Use (O1) and put $x = y = 0$ in (2.1) and subtract the argument of the norm of the resulting inequality from that of inequality (2.1) to get

$$\|F(x + y) + G(x - y) - H(x) - K(y)\| \leq 2\varepsilon. \tag{2.3}$$

Let $\rho^e(x) = \frac{\rho(x)+\rho(-x)}{2}$ and $\rho^o(x) = \frac{\rho(x)-\rho(-x)}{2}$ denote the even and odd parts of a given function ρ, respectively.

If $x \perp y$ then, by (O3), $-x \perp -y$. Hence we can replace x by $-x$ and y by $-y$ in (2.3) to obtain

$$\|F(-x - y) + G(-x + y) - H(-x) - K(-y)\| \leq 2\varepsilon. \tag{2.4}$$

By virtue of triangle inequality and (2.3) and (2.4) we have

$$\|F^o(x+y)+G^o(x-y)-H^o(x)-K^o(y)\| \leq 2\varepsilon, \tag{2.5}$$

$$\|F^e(x+y)+G^e(x-y)-H^e(x)-K^e(y)\| \leq 2\varepsilon, \tag{2.6}$$

for all $x, y \in \mathcal{X}$.

Step (I). Approximating F^o

Let $x \perp y$. Then $y \perp x$, and by (2.5)

$$\|F^o(x+y)-G^o(x-y)-H^o(y)-K^o(x)\| \leq 2\varepsilon. \tag{2.7}$$

It follows from (2.5) and (2.7) that

$$\begin{aligned} & \|2F^o(x+y)-H^o(x)-K^o(x)-H^o(y)-K^o(y)\| \\ \leq\ & \|F^o(x+y)+G^o(x-y)-H^o(x)-K^o(y)\| \\ & +\|F^o(x+y)-G^o(x-y)-H^o(y)-K^o(x)\| \\ \leq\ & 4\varepsilon. \end{aligned} \tag{2.8}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. In particular, for arbitrary x and $y=0$ we get

$$\|2F^o(x)-H^o(x)-K^o(x))\| \quad \leq \quad 4\varepsilon. \tag{2.9}$$

By (2.8) and (2.9), we have

$$\begin{aligned} \|F^o(x+y)-F^o(x)-F^o(y)\| \quad \leq \quad & \frac{1}{2}\|2F^o(x+y)-H^o(x)-K^o(x)-H^o(y)-K^o(y)\| \\ & +\frac{1}{2}\|2F^o(x)-H^o(x)-K^o(x))\| \\ & +\frac{1}{2}\|2F^o(y)-H^o(y)-K^o(y))\| \\ \leq \quad & 6\varepsilon \end{aligned} \tag{2.10}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$.

Given $x \in \mathcal{X}$, by (O4), there exists $y_0 \in \mathcal{X}$ such that $x \perp y_0$ and $x+y_0 \perp x-y_0$. Replacing x and y by $x+y_0$ and $x-y_0$ in (2.10), we have

$$\|F^o(2x)-F^o(x+y_0)-F^o(x-y_0)\| \leq 6\varepsilon. \tag{2.11}$$

Since $x \perp y_0$ and $x \perp -y_0$, it follows from (2.10) that

$$\|F^o(x+y_0) - F^o(x) - F^o(y_0)\| \le 6\varepsilon, \tag{2.12}$$

and

$$\|F^o(x-y_0) - F^o(x) + F^o(y_0)\| \le 6\varepsilon. \tag{2.13}$$

By (2.11), (2.12) and (2.13),

$$\begin{aligned}
\|\frac{1}{2}F^o(2x) - F^o(x)\| &\le \frac{1}{2}\|F^o(2x) - F^o(x+y_0) - F^o(x-y_0)\| \\
&\quad + \frac{1}{2}\|F^o(x+y_0) - F^o(x) - F^o(y_0)\| \\
&\quad + \frac{1}{2}\|F^o(x-y_0) - F^o(x) + F^o(y_0)\| \\
&\le 9\varepsilon.
\end{aligned}$$

Hence $d(F^o, J_{1/2}F^o) \le 9\varepsilon < \infty$. Using the fixed point alternative we conclude the existence of a mapping $R : \mathcal{X} \to \mathcal{Y}$ such that R is a fixed point of $J_{1/2}$ that is $R(2x) = 2R(x)$ for all $x \in \mathcal{X}$. Since $\lim_{n\to\infty} d(J_{1/2}^n F^o, R) = 0$ we easily deduce that $\lim_{n\to\infty} \frac{F^o(2^n x)}{2^n} = R(x)$ for all $x \in \mathcal{X}$.

Indeed, the mapping R is the unique fixed point of $J_{1/2}$ in the set $Y = \{\varphi \in \mathcal{E} : d(F^o, \varphi) < \infty\}$. Hence R is the unique fixed point of $J_{1/2}$ such that $\|F^o(x) - R(x)\| \le K$ for some $K > 0$ and for all $x \in \mathcal{X}$. Again, by applying the fixed point alternative theorem we obtain

$$d(F^o, R) \le 2d(F^o, J_{1/2}F^o) \le 18\varepsilon.$$

Thus

$$\|F^o(x) - R(x)\| \le 18\varepsilon, \tag{2.14}$$

for all $x \in \mathcal{X}$. Let $x \perp y$ and n be a positive integer. Then $2^n x \perp 2^n y$ and so we can replace x and y in (2.10) by $2^n x$ and $2^n y$, respectively. Dividing the both sides by 2^n and letting n tend to ∞ we infer that $R(x+y) = R(x) + R(y)$. Hence R is orthogonally additive.

Step (II). Approximating G^o

Let $x \perp y$. Then $x \perp -y$ and (2.5) yields the following.

$$\|G^o(x+y) + F^o(x-y) - H^o(x) - (-K)^o(y)\| \le 2\varepsilon.$$

Using the same argument as in Step (I), we conclude that there exists a unique orthogonally additive mapping $R' : \mathcal{X} \to \mathcal{Y}$ such that

$$\|G^o(x) - R'(x)\| \leq 18\varepsilon. \tag{2.15}$$

Step (III). Approximating L^o

Using 2.9 we get

$$\|F^o(x) - L^o(x)\| \leq 2\varepsilon, \tag{2.16}$$

so ,by (2.14),

$$\|L^o(x) - R(x)\| \leq \|F^o(x) - L^o(x)\| + \|F^o(x) - R(x)\| \leq 2\varepsilon + 18\varepsilon = 20\varepsilon. \tag{2.17}$$

Step (IV). Approximating G^e

Now we use inequality (2.6) concerning the even parts. Let $x \perp y$. Then $y \perp x$ and by (2.6) we get

$$\|F^e(x+y) + G^e(x-y) - H^e(y) - K^e(x)\| \leq 2\varepsilon. \tag{2.18}$$

By (2.6) and (2.18) we infer that

$$\|F^e(x+y) + G^e(x-y) - L^e(x) - L^e(y)\| \leq 2\varepsilon, \tag{2.19}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. In particular, it follows from $x \perp 0$ that

$$\|F^e(x) + G^e(x) - L^e(x)\| \leq 2\varepsilon, \tag{2.20}$$

for all $x \in \mathcal{X}$. Applying (2.19) and (2.20) we get

$$\begin{aligned}
& \|\big(F^e(x+y) - F^e(x) - F^e(y)\big) + \big(G^e(x-y) - G^e(x) - G^e(y)\big)\| \\
\leq\ & \|F^e(x+y) + G^e(x-y) - L^e(x) - L^e(y)\| \\
& + \|F^e(x) + G^e(x) - L^e(x)\| + \|F^e(y) + G^e(y) - L^e(y)\| \\
\leq\ & 6\varepsilon, \qquad (2.21)
\end{aligned}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$.

Given $x \in \mathcal{X}$, by (O4), there exists $y_0 \in \mathcal{X}$ such that $x \perp y_0$ and $x + y_0 \perp x - y_0$. Hence, by (O3), $x \perp -y_0$ $x + y_0 \perp y_0 - x$ and so, by repeatedly applying (2.21), we get

$$\begin{aligned} &\|\big(F^e(x+y_0) - F^e(x) - F^e(y_0)\big) + \big(G^e(x-y_0) - G^e(x) - G^e(y_0)\big)\| \\ &\leq\ 6\varepsilon, \end{aligned} \tag{2.22}$$

$$\begin{aligned} &\|\big(F^e(x-y_0) - F^e(x) - F^e(y_0)\big) + \big(G^e(x+y_0) - G^e(x) - G^e(y_0)\big)\| \\ &\leq\ 6\varepsilon, \end{aligned} \tag{2.23}$$

$$\begin{aligned} &\|\big(F^e(2y_0) - F^e(x+y_0) - F^e(x-y_0)\big) + \big(G^e(2x) - G^e(x+y_0) - G^e(x-y_0)\big)\| \\ &\leq\ 6\varepsilon. \end{aligned} \tag{2.24}$$

By (O3), $\frac{x+y_0}{2} \perp \pm\frac{x-y_0}{2}$ and so by using (2.21), we obtain

$$\begin{aligned} &\|\big(F^e(x) - F^e(\frac{x+y_0}{2}) - F^e(\frac{x-y_0}{2})\big) + \big(G^e(y_0) - G^e(\frac{x+y_0}{2}) - G^e(\frac{x-y_0}{2})\big)\| \\ &\leq\ 6\varepsilon, \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} &\|\big(F^e(y_0) - F^e(\frac{x+y_0}{2}) - F^e(\frac{x-y_0}{2})\big) + \big(G^e(x) - G^e(\frac{x+y_0}{2}) - G^e(\frac{x-y_0}{2})\big)\| \\ &\leq\ 6\varepsilon. \end{aligned} \tag{2.26}$$

It follows from (2.25) and (2.26) we infer that

$$\|(F^e(x) - F^e(y_0)) - (G^e(x) - G^e(y_0))\| \leq 12\varepsilon. \tag{2.27}$$

Using the triangle inequality, we infer from (2.22), (2.23), (2.24) and (2.27) that

$$\|(F^e(2y_0) - 4F^e(y_0)) + (G^e(2x) - 4G^e(x)\| \leq 42\varepsilon. \tag{2.28}$$

So far, we do not use (2.2). Now we may apply (2.2) and (2.28) to get

$$\begin{aligned} \|\frac{1}{4}G^e(2x) - G^e(x)\| &\leq \frac{1}{4}\|F^e(2y_0) - 4F^e(y_0)\| + \frac{42}{4}\varepsilon \\ &\leq \frac{\varepsilon}{2} + \frac{21\varepsilon}{2} = 11\varepsilon. \end{aligned}$$

Therefore $d(G^e, J_{1/4}G^e) \leq 11\varepsilon < \infty$. Using the fixed point alternative we conclude the existence of a mapping $S' : \mathcal{X} \to \mathcal{Y}$ such that S' is a fixed point of $J_{1/4}$ that is $S'(2x) = 4S'(x)$ for all $x \in \mathcal{X}$. Since $\lim_{n\to\infty} d(J_{1/4}^n G^e, S') = 0$ we easily deduce that $\lim_{n\to\infty} \dfrac{G^e(2^n x)}{2^{2n}} = S'(x)$ for all $x \in \mathcal{X}$.

Indeed, the mapping S' is the unique fixed point of $J_{1/4}$ in the set $Y = \{\psi \in \mathcal{E} : d(G^e, \psi) < \infty\}$. Hence S' is the unique fixed point of $J_{1/4}$ such that $\|G^e(x) - S'(x)\| \leq K$ for some $K > 0$ and for all $x \in \mathcal{X}$. Again, by applying the fixed point alternative theorem we obtain

$$d(G^e, S') \leq \frac{4}{3} d(G^e, J_{1/4}G^e) \leq \frac{44}{3}\varepsilon.$$

Thus

$$\|G^e(x) - S'(x)\| \leq \frac{44}{3}\varepsilon. \tag{2.29}$$

Let $x \perp y$ and n be a positive integer. Then $2^n x \perp 2^n y$ and so we can replace x and y in (2.10) by $2^n x$ and $2^n y$, respectively. Dividing the both sides by 2^{2n} and taking the limit as $n \to \infty$ we infer that $S'(x+y) = S'(x) + S'(y)$. Hence S' is orthogonally additive.

Step (V). Approximating F^e

Let $x \perp y$. Then $x \perp -y$ and (2.6) yields the following.

$$\|G^e(x+y) + F^e(x-y) - H^e(x) - K^e(y)\| \leq 2\varepsilon.$$

By (2.28), we have

$$\|G^e(2x) - 4G^e(x)\| \;\leq\; \|F^e(2y_0) - 4F^e(y_0)\| + 42\varepsilon \leq 44\varepsilon. \tag{2.30}$$

Using the same argument as in Step (IV) and noting to (2.30), we conclude the existence of a unique orthogonally additive mapping $S : \mathcal{X} \to \mathcal{Y}$ such that $S(x) = \lim_{n\to\infty} \frac{F^e(2^n x)}{2^{2n}}$ and

$$\|F^e(x) - S(x)\| \leq \frac{86}{3}\varepsilon. \tag{2.31}$$

Step (VI). Approximating L^e

Inequalities (2.20), (2.29) and (2.31) yield the following.

$$\begin{aligned}\|L^e(x)-S(x)-S'(x)\| &\leq \|F^e(x)+G^e(x)-L^e(x)\|+\|F^e(x)-S(x)\|-\|G^e(x)-S'(x)\|\\ &\leq 2\varepsilon+\frac{86}{3}+\frac{44}{3}\varepsilon\varepsilon\\ &= \frac{136}{3}\varepsilon. \qquad (2.32)\end{aligned}$$

Step (VII). Approximating $f, g, h+k$

Put $T(x)=R(x)+S(x), T'(x)=R'(x)+S'(x)$ and $T''(x)=2R(x)+2S(x)+2S'(x)$. Then T, T' and T'' are orthogonally additive and (2.14), (2.15), (2.17), (2.29), (2.31) and (2.32) yield the following inequalities for each $x\in\mathcal{X}$:

$$\|f(x)-f(0)-T(x)\|\leq\|F^o(x)-R(x)\|+\|F^e(x)-S(x)\|\leq 18\varepsilon+\frac{86}{3}\varepsilon=\frac{140}{3}\varepsilon,$$

$$\|g(x)-g(0)-T'(x)\|\leq\|G^o(x)-R'(x)\|+\|G^e(x)-S'(x)\|\leq 18\varepsilon+\frac{44}{3}\varepsilon=\frac{98}{3}\varepsilon,$$

$$\begin{aligned}\|h(x)+k(x)-h(0)-k(0)-T''(x)\| &\leq 2\|L^o(x)-R(x)\|+2\|L^e(x)-S(x)-S'(x)\|\\ &\leq 40\varepsilon+\frac{136}{3}\varepsilon\\ &= \frac{256}{3}\varepsilon.\end{aligned}$$

Step (VIII). Necessity

Let T be an orthogonally additive mapping such that $\|f(x)-T(x)\|\trianglelefteq\varepsilon$. Then $\|f^e(x)-T^e(x)\|\trianglelefteq\varepsilon$. Note that T^e is an orthogonally additive mapping.

Let $x\in\mathcal{X}$. Using (O4), there exists a vector $y_0\in\mathcal{X}$ such that $x\perp y_0$ and $x+y_0\perp x-y_0$. Then, by (O3), $\frac{x}{2}\perp\frac{y_0}{2}$, $\frac{x+y_0}{2}\perp\frac{x-y_0}{2}$ and $x+y_0\perp y_0-x$. Hence

$$\begin{aligned}T(x) &= T(\frac{x+y_0}{2}+\frac{x-y_0}{2})=T(\frac{x+y_0}{2})+T(\frac{x-y_0}{2})\\ &= T(\frac{x}{2})+T(\frac{y_0}{2})+T(\frac{x}{2})+T(\frac{-y_0}{2})=2T(\frac{x}{2})+2T(\frac{y_0}{2}),\end{aligned}$$

$$\begin{aligned}T(y_0) &= T(\frac{y_0+x}{2}+\frac{y_0-x}{2})=T(\frac{y_0+x}{2})+T(\frac{y_0-x}{2})\\ &= T(\frac{y_0}{2})+T(\frac{x}{2})+T(\frac{y_0}{2})+T(\frac{-x}{2})=2T(\frac{y_0}{2})+2T(\frac{x}{2}),\end{aligned}$$

$$\begin{aligned} T(2x) &= T((x+y_0)+(x-y_0)) = T(x+y_0)+T(x-y_0) \\ &= T(x)+T(y_0)+T(x)+T(-y_0) = 2T(x)+2T(y_0) = 4T(x), \end{aligned}$$

and so $T^e(2x) = 4T^e(x)$. Therefore,

$$\begin{aligned} \|f(2x)-f(-2x)-4f(x)-4f(-x)\| &\leq \|f^e(2x)-4f^e(x)\| \\ &= \|f^e(2x)-T^e(2x)\| + \|4T^e(x)-4f^e(x)\| \\ &\trianglelefteq \varepsilon. \end{aligned}$$

$\square$

Remark 2.3. Let the binary relation $\perp'$ is defined by

$$x \perp' y \Leftrightarrow (x \perp y \text{ or } y \perp x)$$

Then clearly $\perp'$ is a symmetric orthogonality in the sense of Rätz. If f, g, h, k are even mappings, then (2.19) shows that if (2.1) holds for all $x, y \in \mathcal{X}$ with $x \perp y$, then the same holds for all $x, y \in \mathcal{X}$ with $x \perp' y$. Now if T is an orthogonally additive mapping with respect to $\perp'$ then it is trivially an orthogonally additive mapping with respect to $\perp$. To prove the theorem therefore, in the case that all mappings are even, we may omit the assumption that $\perp$ is symmetric.

Remark 2.4. In 1985, Rätz (cf. Corollary 7 of [25]) stated that if $(Y, +)$ is uniquely 2-divisible (i.e. the mapping $\omega : Y \to Y, \omega(y) = 2y$ is bijective), in particular Y is a vector space, then every orthogonally additive mapping T has the form $T = A + P$ with A additive and P quadratic.

The first corollary gives us a sufficient and necessary condition to approximate an orthogonally quadratic mapping by orthogonally additive and orthogonally quadratic mappings.

Corollary 2.5. *Suppose that $\mathcal{X}$ is a real orthogonality space with a symmetric orthogonal relation $\perp$ and $\mathcal{Y}$ is a Banach space. Let $Q : \mathcal{X} \to \mathcal{Y}$ be an orthogonally quadratic mapping. Then a necessary and sufficient condition for the existence of an additive mapping A and an quadratic mapping P with*

$$\|Q(x) - A(x) - P(x)\| \trianglelefteq \varepsilon,$$

is that

$$\|Q(2x) - 4Q(x)\| \trianglelefteq \varepsilon.$$

Proof. Set $f = g = Q$ and $h = k = 2Q$ in Theorem 2.1. Then, by remark 2.3, there exist an additive mapping A and an quadratic mapping P such that

$$\|Q(x) - A(x) - P(x)\| \trianglelefteq \varepsilon.$$

Conversely, if there exists the orthogonally additive mapping $T = A+P$ such that $\|Q(x)-T(x)\| \trianglelefteq \varepsilon$, then the computations in the Step (VIII) of Theorem 2.1 gives rise

$$\|Q(2x) - 4Q(x)\| \trianglelefteq \varepsilon.$$

Note that Q is orthogonally quadratic and so is clearly even, i.e. $Q^e = Q$. □

The second corollary gives a solution of the stability of Pexiderized Cauchy equation (see also [21]).

Corollary 2.6. *Suppose that $\mathcal{X}$ is a real orthogonality space with a symmetric orthogonal relation $\perp$ and $\mathcal{Y}$ is a Banach space. Let the mappings $f, h, k : \mathcal{X} \to \mathcal{Y}$ satisfy the following inequality*

$$\|f(x + y) - h(x) - k(y)\| \leq \varepsilon$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - f(0) - T(x)\| \leq 32\varepsilon$$

$$\|h(x) + k(x) - h(0) - k(0) - 2T(x)\| \leq 16\varepsilon$$

for all $x \in \mathcal{X}$.

Proof. The proof of Step (IV) of Theorem 2.1 states that the condition

$$\|f(2x) - f(-2x) - 4f(x) - 4f(-x)\| \leq \trianglelefteq \varepsilon$$

holds if and only if so does

$$\|g(2x) - g(-2x) - 4g(x) - 4g(-x)\| \leq \trianglelefteq \varepsilon.$$

Hence we may let $G = 0$ in Theorem 2.2. Then $R' = S' = 0$ and the constructions in (2.28) and (2.32) of the proof of the theorem allow us to have $\|F^e(x) - S(x)\| \leq 14\varepsilon$ and $\|L^e(x) - S(x)\| \leq 16\varepsilon$. Then

$$\|f(x) - f(0) - T(x)\| \leq \|F^o(x) - R(x)\| + \|F^e(x) - S(x)\| \leq 18\varepsilon + 14\varepsilon = 32\varepsilon,$$

and

$$\begin{aligned} \|h(x) + k(x) - h(0) - k(0) - 2T(x)\| &\leq 2\|L^o(x) - R(x)\| + 2\|L^e(x) - S(x)\| \\ &\leq 40\varepsilon + 32\varepsilon \\ &= 72\varepsilon. \end{aligned}$$

for all $x \in \mathcal{X}$. □

The third corollary concerns the case that $\mathcal{X}$ is assumed to be an ordinary inner product space.

Corollary 2.7. *Suppose that $\mathcal{H}$ is a real inner product space of dimension greater than or equal 3 and $\mathcal{Y}$ is a Banach space. Let the mappings $f, g, h, k : \mathcal{H} \to \mathcal{Y}$ satisfy the following inequalities*

$$\|f(x+y) + g(x-y) - h(x) - k(y)\| \leq \varepsilon,$$

and

$$\|f(2x) - f(-2x) - 4f(x) - 4f(-x)\| \leq \varepsilon,$$

for all $x, y \in \mathcal{H}$ with $x \perp y$. Then there exist orthogonally additive mappings $T, T', T'' : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{140}{3}\varepsilon,$$

$$\|g(x) - g(0) - T'(x)\| \leq \frac{98}{3}\varepsilon,$$

$$\|h(x) + k(x) - h(0) - k(0) - T''(x)\| \leq \frac{256}{3}\varepsilon,$$

for all $x \in \mathcal{H}$.

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